MECHATRONIC DESIGN

Lab 1: Introduction to Finite Difference (FD) method

The goal of the first laboratory is to approximate solutions to differential equations, i.e., to find a function (or some discrete approximation to this function) that satisfies a given relationship between various of its derivatives on some given region of space and/or time, along with some boundary conditions along the edges of this domain. In general, this is a difficult problem, and only rarely can an analytic formula be found for the solution. **A finite difference method proceeds by replacing the derivatives in the differential equations with finite difference approximations.** This gives a large but finite algebraic system of equations to be solved in place of the differential equation, something that can be done on a computer.

Before tackling this problem, we first consider the more basic question of how we can approximate the derivatives of a known function by finite difference formulas based only on values of the function itself at discrete points. Besides providing a basis for the later development of finite difference methods for solving differential equations, this allows us to investigate several key concepts such as the *order of accuracy* of an approximation in the simplest possible setting.

Let $u(x)$ represent a function of one variable that, unless otherwise stated, will always be assumed to be smooth, meaning that we can differentiate the function several times and each derivative is a well-defined bounded function over an interval containing a particular point of interest $\bar{x}$.

Suppose we want to approximate $\frac{\partial u(\bar{x})}{\partial \bar{x}}$ by a finite difference approximation based only on values of u at a finite number of points near $\bar{x}$. One obvious choice would be to use

$$\frac{\partial \boldsymbol{u}(\overline{\boldsymbol{x}})}{\partial \overline{\boldsymbol{x}}}_{+} \approx \frac{\boldsymbol{u}(\overline{\boldsymbol{x}}+\Delta \boldsymbol{x})-\boldsymbol{u}(\overline{\boldsymbol{x}})}{\Delta \boldsymbol{x}} \tag{1}$$

for some small value of *Δx*. Note that $\frac{\partial \boldsymbol{u}(\overline{\boldsymbol{x}})}{\partial \overline{\boldsymbol{x}}}_{+}$ is the slope of the line interpolating *u* at the points $\bar{x}$ and $\bar{x}+\Delta x$ (see Figure 1.)

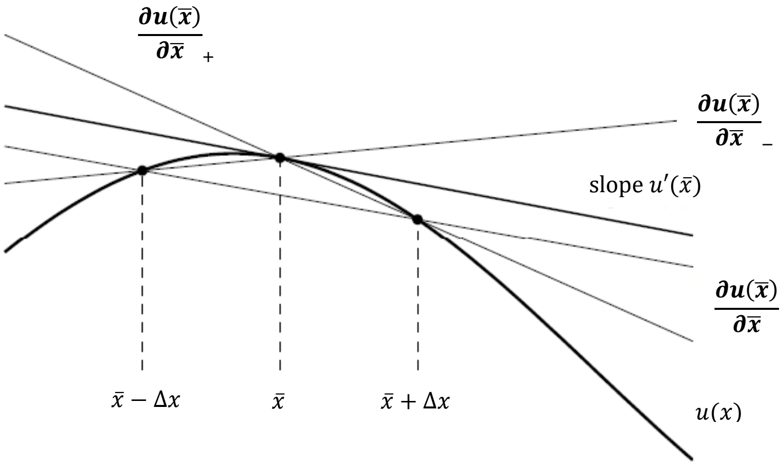


Figure. 1. *Various approximations to* $\frac{\partial u(\bar{x})}{\partial \bar{x}}$ *interpreted as the slope of secant lines.*

Equation (1) is a **one-sided approximation** to $\frac{\partial u(\bar{x})}{\partial \bar{x}}$ since *u(x)* is evaluated only at values of $x \geq \bar{x}$. Another one-sided approximation would be

$$\frac{\partial \boldsymbol{u}(\overline{\boldsymbol{x}})}{\partial \overline{\boldsymbol{x}}}_{-} \approx \frac{\boldsymbol{u}(\overline{\boldsymbol{x}})-\boldsymbol{u}(\overline{\boldsymbol{x}}-\Delta \boldsymbol{x})}{\Delta \boldsymbol{x}}. \tag{2}$$

Each of these formulas gives a *first order accurate* approximation to $u'(\bar{x})$, meaning that the size of the error is roughly proportional to Δx itself.

Another possibility is to use the ***centered approximation***

$$\frac{\partial \boldsymbol{u}(\bar{\boldsymbol{x}})}{\partial \bar{\boldsymbol{x}}} \approx \frac{\boldsymbol{u}(\bar{\boldsymbol{x}}+\Delta \boldsymbol{x})-\boldsymbol{u}(\bar{\boldsymbol{x}}-\Delta \boldsymbol{x})}{\mathbf{2}\Delta \boldsymbol{x}} \tag{3}$$

Equation (3) is the slope of the line interpolating u at $\bar{x}-\Delta x$ and $\bar{x}+\Delta x$ and is simply the average of the two one-sided approximations defined above using Eqs. (1) and (2). **From Figure 1 it should be clear that we would expect $\frac{\partial \boldsymbol{u}(\bar{\boldsymbol{x}})}{\partial \bar{\boldsymbol{x}}}$ to give a better approximation than either of the one-sided approximations.** In fact, this gives a second order accurate approximation - the error is proportional to Δx^2 and hence is much smaller than the error in a first order approximation when Δx is small (Δx < 1).

How are the above finite difference approximations and truncation errors analyzed? The standard approach is to expand each of the function values of u in a Taylor series about the point $\bar{x}$, e.g.,

$$u(\bar{x}+\Delta x)=u(\bar{x})+\Delta x\cdot\frac{\partial u(\bar{x})}{\partial \bar{x}}+\frac{1}{2}\Delta x^2\cdot\frac{\partial^2 u(\bar{x})}{\partial \bar{x}^2}+\frac{1}{6}\Delta x^3\cdot\frac{\partial^3 u(\bar{x})}{\partial \bar{x}^3}+O(\Delta x^4) \tag{4}$$

$$u(\bar{x}-\Delta x)=u(\bar{x})-\Delta x\cdot\frac{\partial u(\bar{x})}{\partial \bar{x}}+\frac{1}{2}\Delta x^2\cdot\frac{\partial^2 u(\bar{x})}{\partial \bar{x}^2}-\frac{1}{6}\Delta x^3\cdot\frac{\partial^3 u(\bar{x})}{\partial \bar{x}^3}+O(\Delta x^4) \tag{5}$$

Using Equation (4) allows us to compute that

$$\left.\frac{\partial \boldsymbol{u}(\bar{\boldsymbol{x}})}{\partial \bar{\boldsymbol{x}}}\right._{+} \approx \frac{\boldsymbol{u}(\bar{\boldsymbol{x}}+\Delta \boldsymbol{x})-\boldsymbol{u}(\bar{\boldsymbol{x}})}{\Delta \boldsymbol{x}} \approx \frac{\partial u(\bar{x})}{\partial \bar{x}}+\frac{1}{2}\Delta x\cdot\frac{\partial^2 u(\bar{x})}{\partial \bar{x}^2}+\frac{1}{6}\Delta x^2\cdot\frac{\partial^3 u(\bar{x})}{\partial \bar{x}^3}+O(\Delta x^3) \tag{6}$$

Recall that $\bar{x}$ is a fixed point so that $\frac{\partial^2 u(\bar{x})}{\partial \bar{x}^2}$, $\frac{\partial^3 u(\bar{x})}{\partial \bar{x}^3}$, etc., are fixed constants independent of Δx. They depend on u of course, but the function is also fixed as we vary Δx.

For Δx sufficiently small, the error will be dominated by the first term $\frac{1}{2}\Delta x\cdot\frac{\partial^2 u(\bar{x})}{\partial \bar{x}^2}$ and all the other terms will be negligible compared to this term, so we expect the error to behave roughly like a constant times Δx, where the constant has the value $\frac{1}{2}\frac{\partial^2 u(\bar{x})}{\partial \bar{x}^2}$.

Similarly, from Equation (5) we can compute that the error in $D_{-}u(\bar{x})$ is

$$\left.\frac{\partial \boldsymbol{u}(\bar{\boldsymbol{x}})}{\partial \bar{\boldsymbol{x}}}\right._{-} \approx \frac{\boldsymbol{u}(\bar{\boldsymbol{x}})-\boldsymbol{u}(\bar{\boldsymbol{x}}-\Delta \boldsymbol{x})}{\Delta \boldsymbol{x}} \approx \frac{\partial u(\bar{x})}{\partial \bar{x}}+\frac{1}{2}\Delta x\cdot\frac{\partial^2 u(\bar{x})}{\partial \bar{x}^2}-\frac{1}{6}\Delta x^2\cdot\frac{\partial^3 u(\bar{x})}{\partial \bar{x}^3}+O\left(\Delta x^3\right). \tag{7}$$

Subtracting Equation (5) from (4) gives

$$u(\bar{x}+\Delta x)-u(\bar{x}-\Delta x)=2\Delta x\cdot\frac{\partial u(\bar{x})}{\partial \bar{x}}+\frac{1}{3}\Delta x^3\cdot\frac{\partial^3 u(\bar{x})}{\partial \bar{x}^3}(\bar{x})+O(\Delta x^5) \tag{8}$$

so the central difference approximation is

$$\frac{\partial \boldsymbol{u}(\bar{\boldsymbol{x}})}{\partial \bar{\boldsymbol{x}}} \approx \frac{\boldsymbol{u}(\bar{\boldsymbol{x}}+\Delta \boldsymbol{x})-\boldsymbol{u}(\bar{\boldsymbol{x}}-\Delta \boldsymbol{x})}{\mathbf{2}\Delta \boldsymbol{x}} \approx \frac{\partial u(\bar{x})}{\partial \bar{x}}+\frac{1}{6}\Delta x^2\cdot\frac{\partial^3 u(\bar{x})}{\partial \bar{x}^3}+O(\Delta x^4) \tag{9}$$

This confirms the second order accuracy of this approximation. Note that all the odd order terms drop out of the Taylor series expansion (9) for $\frac{\partial u(\bar{x})}{\partial \bar{x}}$. This is typical with centered approximations and typically leads to a <u>higher order approximation</u>.

Approximations to the second order derivative $\frac{\partial^2 u(\bar{x})}{\partial^2 \bar{x}}$ can be obtained in an analogous manner (using Taylor series) leading to

$$\frac{\partial^2 u(\bar{x})}{\partial \bar{x}^2} \approx \frac{\boldsymbol{u}(\bar{\boldsymbol{x}}+\Delta \boldsymbol{x})-\mathbf{2}\boldsymbol{u}(\bar{\boldsymbol{x}})+\boldsymbol{u}(\bar{\boldsymbol{x}}-\Delta \boldsymbol{x})}{\Delta \boldsymbol{x}^{\mathbf{2}}}+O(\Delta x^2) \tag{10}$$

The second order mixed derivative $\frac{\partial^2 u(\bar{x},\bar{y})}{\partial\bar{x}\partial\bar{y}}$ is:

$$\frac{\partial^2 u(\bar{x},\bar{y})}{\partial\bar{x}\partial\bar{y}} \approx \frac{u(\bar{x}+\Delta x,\bar{y}+\Delta y) - u(\bar{x}-\Delta x,\bar{y}+\Delta y) - u(\bar{x}+\Delta x,\bar{y}-\Delta y) + u(\bar{x}-\Delta x,\bar{y}-\Delta y)}{4\Delta x\Delta y} + O(\Delta x^2,\Delta y^2)$$

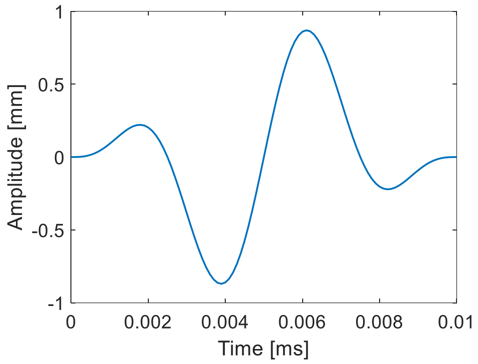
Tasks to do:

<u>Task 1:</u>

Solve a one-dimensional acoustic wave equation using finite difference method for an **aluminum/copper/steel** rod. Implement a source code in MATLAB to obtain wave displacement in time and space domains. The one-dimensional wave equation is:

$$\begin{gathered}\frac{\partial^2 u(x,t)}{\partial t^2} = c^2\frac{\partial^2 u(x,t)}{\partial x^2}, \\ c = \sqrt{\frac{E}{\rho}}, \qquad 0 \le x \le L, \qquad t \ge 0\end{gathered} \tag{11}$$

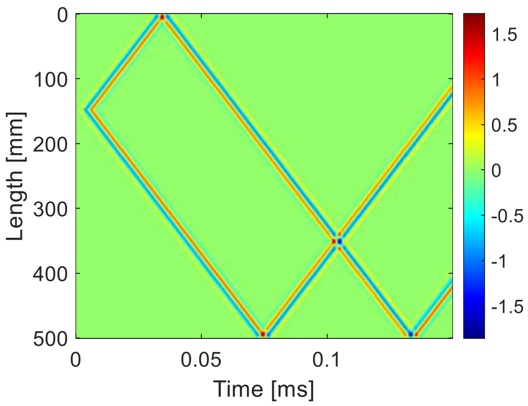
where E is Young's modulus, ρ is mass density, L is the length of the rod.
Use the following constant parameters for your calculations: **Δx = 1 mm, Δt = 0.12 μs**. The excitation signal is a 200 kHz two cycle sine wave modulated by a Hanning window:



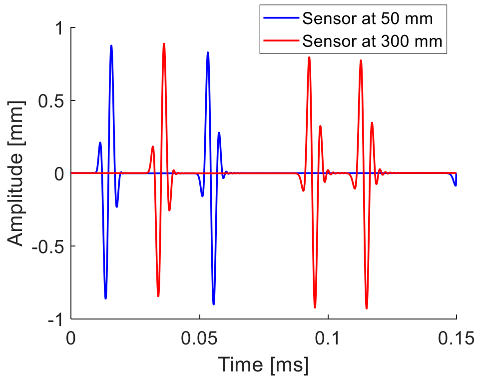
Calculate the wave speed of the simulated wave motion.

Exemplary results:

A spatio-temporal acoustic wave motion data with an excitation located at 150 mm:



Temporal acoustic wave motion data measured at 50 and 300 mm:



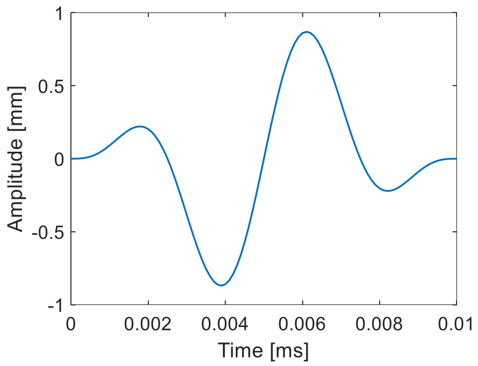
Task 2:

Solve a two-dimensional acoustic wave equation using finite difference method for an **aluminum** plate. Implement a source code in MATLAB to obtain wave displacement in time and space domains. The two-dimensional acoustic wave equation is:

$$\frac{\partial^2 u(x,y,t)}{\partial t^2} = c^2 \left(\frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2} \right), \qquad (12)$$
$$c = \sqrt{\frac{E}{\rho}}, \qquad 0 \le x \le L_1, \qquad 0 \le y \le L_2, \qquad t \ge 0$$

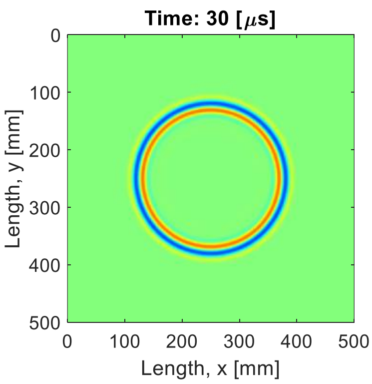
where E is Young's modulus, ρ mass density, L is the length of the rod. Use the following constant parameters for your calculations: **Δx = Δy = 1 mm, Δt = 0.12 μs**.

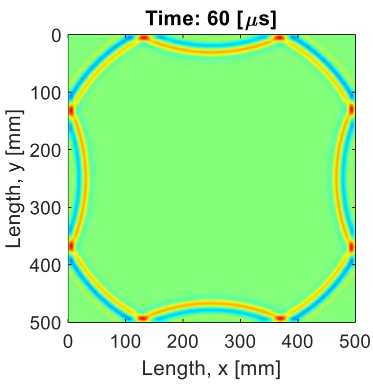
The excitation signal is a 200 Hz two cycle sine wave modulated by a Hanning window:



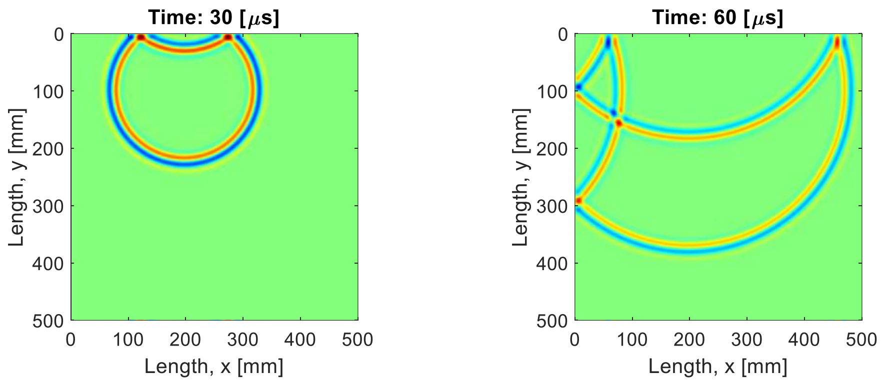
Exemplary results:

Wave motion data for the excitation located in the middle of the plate:



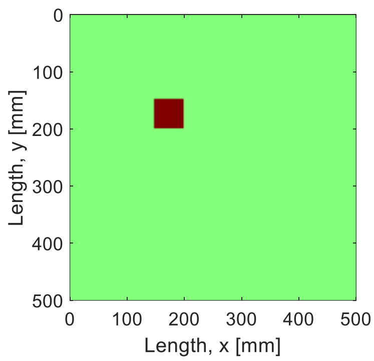


Wave motion data for the excitation located in at the location x = 200 mm, y = 100 mm:

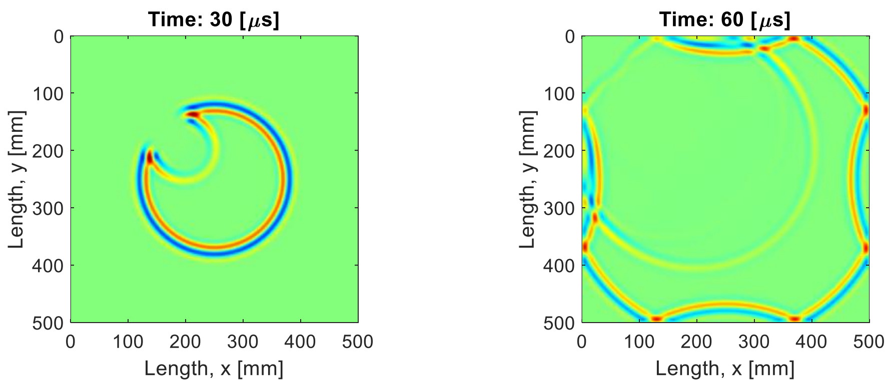


Task 3:

Repeat the simulations from Task 2 with the addition of discontinuity in your model, for example, as:



Exemplary results:



Task 4:

Solve two-dimensional elastodynamic wave equations for an isotropic, linear material using finite difference method for an aluminum plate. Implement a source code in MATLAB to obtain wave displacement in time and space domains.

$$\begin{cases} \rho \dfrac{\partial^2 u_x}{\partial t^2} = \dfrac{\partial \tau_{xx}}{\partial x} + \dfrac{\partial \tau_{xz}}{\partial z} + F_x \\ \rho \dfrac{\partial^2 u_z}{\partial t^2} = \dfrac{\partial \tau_{xz}}{\partial x} + \dfrac{\partial \tau_{zz}}{\partial z} + F_z \end{cases}$$

whereas stress-strain relations are as follow

$$\tau_{xx} = (\lambda + 2\mu)\frac{\partial u_x}{\partial x} + \lambda \frac{\partial u_z}{\partial z}$$

$$\tau_{zz} = (\lambda + 2\mu)\frac{\partial u_z}{\partial z} + \lambda \frac{\partial u_x}{\partial x}$$

$$\tau_{xz} = \mu\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)$$

where

ρ density,

μ shear modulus,

λ first Lamé constant

u displacement vector

F force vector

Exemplary results: